Elastic Vector Wave Field from Cartesian to Curvilinear Coordinate System & its Application to Land Based Unconventional Multicomponent Seismic Imaging, Micro Seismic Forward Modeling and Generalized Wave Field Inversion in Permian Basin,

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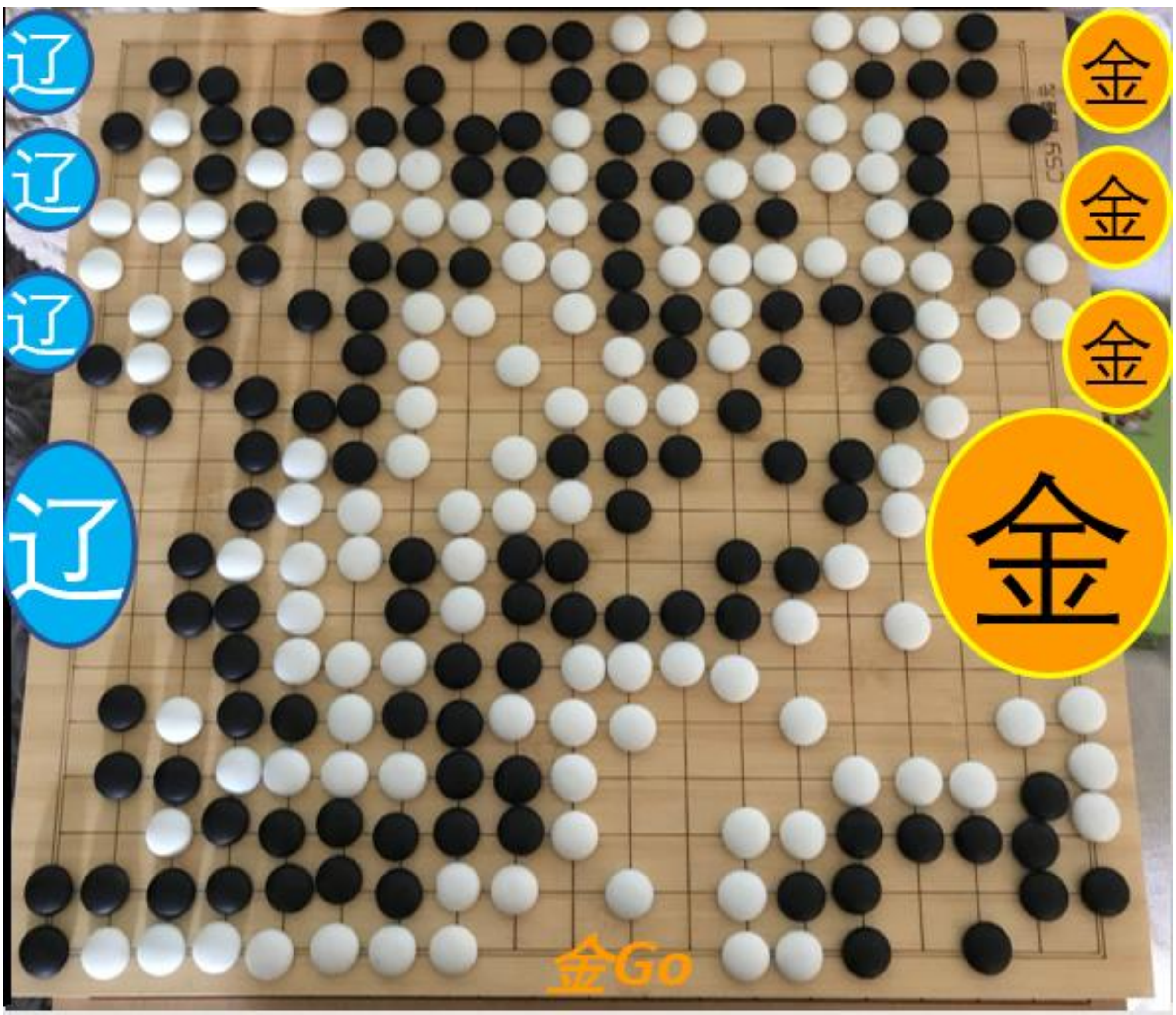

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Chapter 1: Executive Summary for Business Driver

Unconventional technology revolution in hydrocarbon production has raised quite a few challenges in upstream oil industry not only in engineering but also in geosciences due to the nature of the technology in man-made fracture reservoirs and the operation on land terrains. The influence of the revolution on the geophysical chapter has triggered a deep dive in the dynamic elastic wavefield system both on exploration-development geophysics and also on theoretical geophysics because the simplified scalar-based P wave field and its imaging based mainly on traditional signal processing are giving away to multi-components of surface 3C, 9C and borehole micro seismic largely based on theoretical elastic vector wavefield of its tensor nature.

To make things more complicated, under a land undulated topographic surface, not only the normal stress but also the shear stress boundary condition play an important role on the free surface and the surface effect will cascade down to the undulated subsurface deformation which affects drilling, fracking, coupling between nature fracture and hydraulic fracture and finally affects the production.

In short, the above unconventional challenges in geophysics converge to a common call for the Earth dynamic system to be represented by Riemann Space instead of Euclidian Space in order to improve the analysis process of the multicomponent seismic data as well as model-based unconventional seismic reservoir characterization both in seismic lithology as well as in seismic fracture framework so that the unconventional production can be optimized through the geophysical value chains in corporation with other data value chains in petroleum engineering and geosciences.

In one-sentence summary, we need a comprehensive simulator and an inversion kernel for elastic vector wavefield in deformation space under curvilinear coordinates.

Chapter 2: Coordinate Representation in General Curvilinear System

A point P(x_1, x_2, x_3) in space can be represented with a location vector $\boldsymbol{r} = x_i\, \boldsymbol{e}_i$, $i = 1,2,3$. In this document, we will frequently use *Einstein Summation* and will also generalize tensor notation in which a scalar is a zero order tensor, a vector is a first order tensor, a stress or strain and so on is a second order tensor, a Levi-Civita notation ε_{ijk} is a third order tensor, and generalized elastic moduli C_{ijkl} is a forth order tensor.

In Cartesian coordinate, a basis vector $\boldsymbol{e}_i$, $i = 1, 2, 3$, is a constant unit vector and the 3 unit vectors representing 3D space are orthogonal. Therefore they form a global orthogonal basis group in R^3. In general curvilinear coordinates, the basis vectors vary from place to place, they do not have to be unit vectors, and they do not have to be orthogonal. As we know, the good examples are the spherical coordinates and the cylindrical coordinates even though these two curvilinear coordinates are orthogonal as design. The cylindrical coordinates are widely used by petroleum engineers in wellbores.

In our unconventional deformation space with arbitrary undulation of free surface as well as complicated subsurface horizons or corresponding deformed geocells for unconventional model based seismic reservoir characterization in seismic facies and first order seismic defined fracture framework, a fully orthogonal group of bases is hard to exist to help us to simplify the simulation on our observed data of 3C, 9C and micro seismic data. Therefore the orthogonality is not guaranteed due to the scale of complication in deformation space.

In order to differentiate curvilinear coordinates from Cartesian coordinates, we define a general curvilinear coordinate on a point as P(ξ_1, ξ_2, ξ_3). However, we do not know its basis yet in order to define a representation of its location vector, so we need to follow the basic differential geometry rule to define the local bases:

$$\boldsymbol{l}_i = \frac{\partial \boldsymbol{r}}{\partial \xi_i}, \ i = 1, 2, 3 \tag{2.1}$$

Define the norm of the curvilinear local bases:

$$l_i = |\,\boldsymbol{l}_i| = \left|\frac{\partial \boldsymbol{r}}{\partial \xi_i}\right| \tag{2.2}$$

Normalize the bases:

$$\boldsymbol{b}_i = \frac{\boldsymbol{l}_i}{l_i} \tag{2.3}$$

Representation of location vector in curvilinear coordinates at point P(ξ_1, ξ_2, ξ_3):

$$\boldsymbol{r} = \xi_i\, \boldsymbol{l}_i = \xi_i\, l_i\, \boldsymbol{b}_i \tag{2.4}$$

Chapter 3: *Jacobian* Matrix and its Transformation

Define coordinate functionals between *Cartesian* and Curvilinear:

$$x_1 = f_1(\xi_1, \xi_2, \xi_3),\ x_2 = f_2(\xi_1, \xi_2, \xi_3),\ x_3 = f_3(\xi_1, \xi_2, \xi_3) \tag{3.1}$$

$$\xi_1 = g_1(x_1, x_2, x_3),\ \xi_2 = g_2(x_1, x_2, x_3),\ \xi_3 = g_3(x_1, x_2, x_3) \tag{3.2}$$

Basis vectors from Equation (2.1) can be represented by the above functional relations in partial differential as:

$$\boldsymbol{l}_1 = \frac{\partial \boldsymbol{r}}{\partial \xi_1} = \frac{\partial \boldsymbol{r}}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial \boldsymbol{r}}{\partial x_2}\frac{\partial x_2}{\partial \xi_1} + \frac{\partial \boldsymbol{r}}{\partial x_3}\frac{\partial x_3}{\partial \xi_1} = \frac{\partial x_1}{\partial \xi_1}\boldsymbol{e}_1 + \frac{\partial x_2}{\partial \xi_1}\boldsymbol{e}_2 + \frac{\partial x_3}{\partial \xi_1}\boldsymbol{e}_3$$

or in tensor notation:

$$\boldsymbol{l}_i = \frac{\partial \boldsymbol{r}}{\partial \xi_i} = \frac{\partial x_j}{\partial \xi_i}\boldsymbol{e}_j = J_{ij}\,\boldsymbol{e}_j \tag{3.3}$$

Where J_{ij} is the second order *Jacobian* tensor:

$$J_{ij} \equiv \frac{\partial x_j}{\partial \xi_i} \tag{3.4}$$

Similarly, the **inverse of the second order *Jacobian* tensor is** $\boldsymbol{\frac{\partial \xi_j}{\partial x_i}}$ **.**

In our unconventional land deformation, we only concern topography and undulated subsurface horizons, so our coordinate system can be simplified as quasi-orthogonal:

$$x_1 = \xi_1,\ x_2 = \xi_2,\ x_3 = f_3(\xi_1, \xi_2, \xi_3) \tag{3.5}$$

$$\xi_1 = x_1,\ \xi_2 = x_2,\ \xi_3 = g_3(x_1, x_2, x_3) \tag{3.6}$$

Therefore in our special case, *Jacobian* becomes:

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} \tag{3.7}$$

For hydrocarbon exploration and development, we start from *Cartesian* coordinates, so sometimes the inverse of *Jacobian* seems more important to us even though with the current great accuracy of satellite positioning system, we are using GPS of spherical coordinates for double checking.

$$\boldsymbol{J^{-1}} = \begin{bmatrix} \frac{\partial \xi_1}{\partial x_1} & \frac{\partial \xi_1}{\partial x_2} & \frac{\partial \xi_1}{\partial x_3} \\ \frac{\partial \xi_2}{\partial x_1} & \frac{\partial \xi_2}{\partial x_2} & \frac{\partial \xi_2}{\partial x_3} \\ \frac{\partial \xi_3}{\partial x_1} & \frac{\partial \xi_3}{\partial x_2} & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial \xi_3}{\partial x_1} & \frac{\partial \xi_3}{\partial x_2} & \frac{\partial \xi_3}{\partial x_3} \end{bmatrix} \tag{3.8}$$

In our special case of land deformation, ξ_3 defines the surfaces of the deformation body and it will also carry downward the free traction surface condition and we can also take air loading as well into the consideration.

Also from the structure of our special localized *Jacobian* matrix, we can see that our curvilinear coordinate system is quasi-orthogonal because the normal vector of the surface for ξ_3 is not required to be perpendicular to the normal vectors of the other two coordinate defined surfaces. Actually for most of our local deformation, the condition of $\xi_3 \perp (\xi_1, \xi_2)$ or (x_1, x_2) does not exist.

Finally, it is a psychological issue that human habit dominates human life. We have gotten so used to Cartesian in *Euclidian* space that we hardly realize that the Euclidian is actually a vacuum space which does not exist. We actually live in Riemann space which is an energy space defined by curvilinear coordinate system. We may be able to cover *Metric Tensor* for more details of energy space in the following chapters.

For our unconventional business, we need to dig into what is happening in Riemann space such as how rock fractures in deformation space, how difficult we are drilling in deformation space, how elastic vector wavefield propagates in deformation space and so on.

Chapter 4: General Elastic Vector Wavefield in General Curvilinear System

Conservation of Momentum (*Equation of Motion*) independent of coordinates:

$$\frac{\partial}{\partial t}\iiint \rho \frac{\partial \boldsymbol{u}}{\partial t}\,dv = \oiint \boldsymbol{T}dS + \iiint \boldsymbol{f}dv \tag{4.1}$$

Using Cauchy Formula:

$$T_i = \tau_{ij}n_j \tag{4.2}$$

$$\boldsymbol{T} = \boldsymbol{\tau}_i\boldsymbol{\cdot n} \tag{4.3}$$

Where n_j is the projection of the directional cosine on arbitrary surfaces of the curvilinear coordinate system, $\boldsymbol{\tau}_i$ is the row vector of the second order stress tensor.

Substitute Equation (4.3) into (4.1):

$$\frac{\partial}{\partial t}\iiint \rho \frac{\partial \boldsymbol{u}}{\partial t}\,dv = \oiint \boldsymbol{T}dS + \iiint \boldsymbol{f}dv = \oiint \boldsymbol{\tau_i} \boldsymbol{\cdot}\, \boldsymbol{n}dS + \iiint \boldsymbol{f}dv$$

$$= \oiint \boldsymbol{\tau_i} \boldsymbol{\cdot}\, d\boldsymbol{S} + \iiint \boldsymbol{f}dv \tag{4.4}$$

Applying *Gaussian* Theorem, Equation (4.4) becomes:

$$\frac{\partial}{\partial t}\iiint \rho \frac{\partial \boldsymbol{u}}{\partial t}\,dv = \iiint \boldsymbol{\nabla} \boldsymbol{\cdot}\ \boldsymbol{\tau_i}\, dv + \iiint \boldsymbol{f}dv \tag{4.5}$$

By removing the volume integral in (4.5), *Conservation of Momentum* is represented by partial differential equation of **vector wavefield** as:

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla} \boldsymbol{\cdot}\ \boldsymbol{\tau_i} + \boldsymbol{f} \tag{4.6}$$

To get the divergence of a vector in the first term on the right of Equation (4.6) for curvilinear coordinates, we need to introduce ***Generalized Riemann Metric Tensor***. Define an infinitesimal arc *ds* to measure Riemann space curved length with curvilinear coordinates. In infinitesimal, the arc vector ***ds*** is approximated by position vector variation ***dr***.

$$ds^2 = d\boldsymbol{s}\boldsymbol{\cdot}d\boldsymbol{s} = d\boldsymbol{r}\boldsymbol{\cdot}d\boldsymbol{r} \tag{4.7}$$

From full differential with Equations (2.1), (3.3) and (3.4), we get

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{r}}{\partial \xi_i}\, d\xi_i = d\xi_i\, \boldsymbol{l}_i = d\xi_i\, \frac{\partial x_m}{\partial \xi_i}\, \boldsymbol{e}_m = d\xi_i\, J_{im}\, \boldsymbol{e}_m \tag{4.8}$$

Similarly as (4.8) in general curvilinear coordinates, (4.7) becomes

$$d\boldsymbol{r}\cdot d\boldsymbol{r} = d\xi_i J_{im} \boldsymbol{e}_m d\xi_j J_{jn} \boldsymbol{e}_n = J_{im} J_{jn} d\xi_i d\xi_j \boldsymbol{e}_m \boldsymbol{e}_n \tag{4.9}$$

Because $\boldsymbol{e}_m$ is *Cartesian* orthogonal basis, $\boldsymbol{e}_m \boldsymbol{e}_n = 0$ if $m \neq n$. Therefore (4.9) becomes:

$$ds^2 = d\boldsymbol{r}\cdot d\boldsymbol{r} = J_{im} J_{jm} d\xi_i d\xi_j \equiv g_{ij} d\xi_i d\xi_j \tag{4.10}$$

Where g_{ij} is called ***Generalized Riemann Metric Tensor***. Rewrite it as:

$$g_{ij} \equiv J_{im} J_{jm} \tag{4.11}$$

As we can see, the metric tensor is derived from *Jacobian* matrix and it dictates the partial differential operators of ***gradience, divergence, and curl*** which are key operators for *Equation of Motion*.

Let's represent divergence in curvilinear system in order to solve Equation (4.6). Assume ***A*** is an arbitrary vector in curvilinear coordinates. The divergence of ***A*** is:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \lim_{v\to 0}\frac{1}{v}\oint A_{\boldsymbol{n}}\, ds \tag{4.12}$$

n denotes the normal direction of surface *s* for infinitesimal volume *v* which is formed by infinitesimal arc *ds* along 3 curvilinear coordinates:

$$v = ds_1 * ds_2 * ds_3 \tag{4.13}$$

Because in Cartesian coordinates, infinitesimal $v = |dx_1|*|dx_2|*|dx_3|$, and in curvilinear coordinates

$$dx_i = \frac{\partial x_i}{\partial \xi_1} d\xi_1 + \frac{\partial x_i}{\partial \xi_2} d\xi_2 + \frac{\partial x_i}{\partial \xi_3} d\xi_3 = \frac{\partial x_i}{\partial \xi_j} d\xi_j = J_{ji}\, d\xi_j \tag{4.14}$$

For the norm of $|dx_i|$

$$dx_1^2 = dx_1\, dx_1 = J_{i1}\, d\xi_i J_{j1}\, d\xi_j = J_{i1} J_{j1}\, d\xi_i d\xi_j$$

$$= \left(\frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2 + \frac{\partial x_1}{\partial \xi_3} d\xi_3\right)\left(\frac{\partial x_1}{\partial \xi_1} d\xi_1 + \frac{\partial x_1}{\partial \xi_2} d\xi_2 + \frac{\partial x_1}{\partial \xi_3} d\xi_3\right)$$

$$|dx_1| = \sqrt{J_{i1} J_{j1}\, d\xi_i d\xi_j} \tag{4.14.1}$$

Similarly:

$$|dx_2| = \sqrt{J_{i2} J_{j2}\, d\xi_i d\xi_j} \tag{4.14.2}$$

$$|dx_3| = \sqrt{J_{i3} J_{j3}\, d\xi_i d\xi_j} \tag{4.14.3}$$

Therefore:

$$v = ds_1 ds_2 ds_3 = |dx_1||dx_2||dx_3| = J_{j1}\, d\xi_j J_{m2}\, d\xi_m J_{n3}\, d\xi_n = J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n \qquad (4.15)$$

Let us review vector $\boldsymbol{A}$ variation across the closed infinitesimal surface *ds* in Equation (4.12). If the close surface has no source, the in-and-out vector flux will be cancellated. For a non-zero sourcing case, in $\xi_1$direction we have by using Equations (4.13) and (4.14):

$$\frac{\partial}{\partial \xi_1}(A_1\, ds_2 ds_3)^*\, d\xi_1 = (A_1\, ds_2 ds_3)_{\text{out}} - (A_1\, ds_2 ds_3)_{\text{in}} = \frac{\partial}{\partial \xi_1}(A_1 J_{j2} J_{m3}\, d\xi_j d\xi_m) d\xi_1 \qquad (4.16)$$

Similarly, we get:

$$\frac{\partial}{\partial \xi_2}(A_2\, ds_1 ds_3)^*\, d\xi_2 = \frac{\partial}{\partial \xi_2}(A_2 J_{j1} J_{m3}\, d\xi_j d\xi_m)\, d\xi_2 \qquad (4.17)$$

$$\frac{\partial}{\partial \xi_3}(A_3\, ds_1 ds_2)^*\, d\xi_3 = \frac{\partial}{\partial \xi_3}(A_3 J_{j1} J_{m2}\, d\xi_j d\xi_m)\, d\xi_3 \qquad (4.18)$$

Substitute Equations (4.15) to (4.18) back to (4.12), we get divergence in non-orthogonal curvilinear coordinates as:

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial \xi_1}\left(A_1 J_{j2} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1 + \frac{\partial}{\partial \xi_2}\left(A_2 J_{j1} J_{m3}\, d\xi_j d\xi_m\right) d\xi_2 + \frac{\partial}{\partial \xi_3}\left(A_3 J_{j1} J_{m2}\, d\xi_j d\xi_m\right) d\xi_3\right] \qquad (4.19)$$

For orthogonal case or quasi-orthogonal case as shown by *Jacobian* matrix Equation (3.7) for our subsurface deformation, the above equation will be much simpler.

In Equation (4.6), for the first term on the right, we have divergence:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau_i} = \boldsymbol{\nabla} \cdot \begin{bmatrix} \boldsymbol{\tau_{11}} & \boldsymbol{\tau_{12}} & \boldsymbol{\tau_{13}} \\ \boldsymbol{\tau_{21}} & \boldsymbol{\tau_{22}} & \boldsymbol{\tau_{23}} \\ \boldsymbol{\tau_{31}} & \boldsymbol{\tau_{32}} & \boldsymbol{\tau_{33}} \end{bmatrix} \qquad (4.20)$$

From General Hooke's Law,

$$\boldsymbol{\tau} = \boldsymbol{C_{ijkl}} \boldsymbol{\epsilon_{kl}} = \boldsymbol{C_{ijkl}} \boldsymbol{u_{k,l}} \qquad (4.21)$$

Therefore from Equation (4.19) to (4.21), we have the divergence on stress tensor as:

$$\boldsymbol{\nabla} \cdot \boldsymbol{\tau_1} = [\tau_{11} \quad \tau_{12} \quad \tau_{13}] = \frac{1}{J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{j2} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1 + \frac{\partial}{\partial \xi_2}\left(\tau_{12} J_{j1} J_{m3}\, d\xi_j d\xi_m\right) d\xi_2 + \frac{\partial}{\partial \xi_3}\left(\tau_{13} J_{j1} J_{m2}\, d\xi_j d\xi_m\right) d\xi_3\right] =$$

$$\frac{1}{J_{j1}J_{m2}J_{n3}\,d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial\xi_1}\left(C_{11kl}u_{k,l}\,J_{j2}J_{m3}\,d\xi_j d\xi_m\right)d\xi_1 + \frac{\partial}{\partial\xi_2}\left(C_{12kl}u_{k,l}\,J_{j1}J_{m3}\,d\xi_j d\xi_m\right)d\xi_2 + \frac{\partial}{\partial\xi_3}\left(C_{13kl}u_{k,l}\,J_{j1}J_{m2}\,d\xi_j d\xi_m\right)d\xi_3\right] \tag{4.22}$$

Similarly**,**

$$\boldsymbol{\nabla}\bullet\,\boldsymbol{\tau_2} = [\tau_{21}\quad \tau_{22}\quad \tau_{23}] = \frac{1}{J_{j1}J_{m2}J_{n3}\,d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial\xi_1}\left(\tau_{21}\,J_{j2}J_{m3}\,d\xi_j d\xi_m\right)d\xi_1 + \frac{\partial}{\partial\xi_2}\left(\tau_{22}\,J_{j1}J_{m3}\,d\xi_j d\xi_m\right)d\xi_2 + \frac{\partial}{\partial\xi_3}\left(\tau_{23}\,J_{j1}J_{m2}\,d\xi_j d\xi_m\right)d\xi_3\right] = \frac{1}{J_{j1}J_{m2}J_{n3}\,d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial\xi_1}\left(C_{21kl}u_{k,l}\,J_{j2}J_{m3}\,d\xi_j d\xi_m\right)d\xi_1 + \frac{\partial}{\partial\xi_2}\left(C_{22kl}u_{k,l}\,J_{j1}J_{m3}\,d\xi_j d\xi_m\right)d\xi_2 + \frac{\partial}{\partial\xi_3}\left(C_{23kl}u_{k,l}\,J_{j1}J_{m2}\,d\xi_j d\xi_m\right)d\xi_3\right] \tag{4.23}$$

$$\boldsymbol{\nabla}\bullet\,\boldsymbol{\tau_3} = [\tau_{31}\quad \tau_{32}\quad \tau_{33}] = \frac{1}{J_{j1}J_{m2}J_{n3}\,d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial\xi_1}\left(\tau_{31}\,J_{j2}J_{m3}\,d\xi_j d\xi_m\right)d\xi_1 + \frac{\partial}{\partial\xi_2}\left(\tau_{32}\,J_{j1}J_{m3}\,d\xi_j d\xi_m\right)d\xi_2 + \frac{\partial}{\partial\xi_3}\left(\tau_{33}\,J_{j1}J_{m2}\,d\xi_j d\xi_m\right)d\xi_3\right] = \frac{1}{J_{j1}J_{m2}J_{n3}\,d\xi_j d\xi_m d\xi_n}\left[\frac{\partial}{\partial\xi_1}\left(C_{31kl}u_{k,l}\,J_{j2}J_{m3}\,d\xi_j d\xi_m\right)d\xi_1 + \frac{\partial}{\partial\xi_2}\left(C_{32kl}u_{k,l}\,J_{j1}J_{m3}\,d\xi_j d\xi_m\right)d\xi_2 + \frac{\partial}{\partial\xi_3}\left(C_{33kl}u_{k,l}\,J_{j1}J_{m2}\,d\xi_j d\xi_m\right)d\xi_3\right] \tag{4.24}$$

Note, $\boldsymbol{\tau_i}$ is expressed as the row vector of stress tensor $\boldsymbol{\tau_{ij}}$ and the index *j* is eaten up by Einstein summation.

The last three equations show the complication of the spatial operator of equation of motion on the displacement of an elastic particle motion vector in deformation space with curvilinear coordinates. However, the equation of motion in time domain is still the same and can be generalized by propagator matrix along time axis. We will discuss this generalization of propagator matrix in next chapter.

Chapter 5: Generalized Propagator Matrix in Time Domain

Define the density normalized spatial partial deferential operator matrix as:

$$\boldsymbol{G} \equiv (\textbf{\textit{Spatial Operator Matrix}})/\rho \tag{5.1}$$

and define the density normalized body force in unit volume as:

$$\boldsymbol{F} \equiv \frac{\boldsymbol{f}}{\rho} \tag{5.2}$$

where ***Spatial Operator Matrix*** is defined by Equations (4.22), (4.23) and (4.24). Therefore *Equation of Motion* defined by Equation (4.6) becomes:

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{G}\boldsymbol{u} + \boldsymbol{F} \tag{5.3}$$

Create a simultaneous equation for propagator vector ***b***

$$\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \boldsymbol{u}}{\partial t}$$

$$\boldsymbol{b} \equiv \begin{bmatrix} \boldsymbol{u} \\ \frac{\partial \boldsymbol{u}}{\partial t} \end{bmatrix} \tag{5.4}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \begin{bmatrix} \frac{\partial \boldsymbol{u}}{\partial t} \\ \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ \frac{\partial \boldsymbol{u}}{\partial t} \end{bmatrix} + \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} \boldsymbol{b} + \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix} \tag{5.5}$$

Define *Propagator Matrix* as:

$$\boldsymbol{P} \equiv \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} \tag{5.6}$$

and the body force related inhomogeneous vector as:

$$\boldsymbol{Б} \equiv \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix} \tag{5.7}$$

Then Equation (5.5) becomes the standard propagator matrix equation:

$$\frac{\partial \boldsymbol{b}}{\partial t} = \boldsymbol{P}\boldsymbol{b} + \boldsymbol{Б} \tag{5.8}$$

Equation (5.8) can be easily solved by：

$$\frac{d\boldsymbol{b}}{\boldsymbol{b}} = (\boldsymbol{P} + \frac{\text{Б}}{\mathbf{b}})dt$$

$\ln(\boldsymbol{b}) = \boldsymbol{P}t + \int_0^t \frac{Б}{\mathbf{b}} dt$

$$\boldsymbol{b} = e^{\boldsymbol{P}t} e^{\int_0^t \frac{Б}{\mathbf{b}} dt} = \Gamma(\mathrm{t})\, e^{\boldsymbol{P}t} \tag{5.9}$$

where

$$\Gamma(\mathrm{t}) \equiv e^{\int_0^t \frac{Б}{\mathbf{b}} dt}$$

Using *Constant to Variable Transformation* in *Mathematical Physics* by substituting (5.9) back to (5.8), we get:

$$\frac{d\boldsymbol{b}}{dt} = \frac{d\Gamma(\boldsymbol{t})}{dt} e^{\boldsymbol{P}t} + \Gamma(\mathrm{t})\, \boldsymbol{P}\, e^{\boldsymbol{P}t} = \boldsymbol{P}\, \Gamma(\mathrm{t})\, e^{\boldsymbol{P}t} + \boldsymbol{Б}$$

$$\frac{d\Gamma(\boldsymbol{t})}{dt} = \boldsymbol{Б}\, e^{-\boldsymbol{P}t}$$

$$\Gamma(\mathrm{t}) = \int_0^t Б(t')e^{-\boldsymbol{P}t'} dt' \tag{5.10}$$

Substitute (5.10) back to (5.9):

$$\boldsymbol{b} = \Gamma(\mathrm{t}) e^{\boldsymbol{P}t} = e^{\boldsymbol{P}t} \int_0^t Б(t')e^{-\boldsymbol{P}t'} dt' = \int_0^t Б(t')e^{\boldsymbol{P}(\boldsymbol{t}-t')} dt' \tag{5.11}$$

Using *Taylor Expansion* for *Propagator Matrix*,

$$e^{\boldsymbol{P}(\boldsymbol{t}-t')} = \boldsymbol{P}^0 + \boldsymbol{P}(t-t') + \frac{\boldsymbol{P}^2}{2!}(t-t')^2 + \frac{\boldsymbol{P}^3}{3!}(t-t')^3 + \ldots\ldots \tag{5.12}$$

From Equation (5.6), we have

$$\boldsymbol{P} \equiv \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix}$$

Therefore,

$$\boldsymbol{P}^0 = \begin{bmatrix} 1 & 0 \\ \mathbf{0} & 1 \end{bmatrix}$$

$$\boldsymbol{P}^2 = \boldsymbol{P}\,\boldsymbol{P} = \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{G} & 0 \\ \mathbf{0} & \boldsymbol{G} \end{bmatrix}$$

$$\boldsymbol{P}^3 = \boldsymbol{P}\,\boldsymbol{P}\,\boldsymbol{P} = \begin{bmatrix} \boldsymbol{G} & 0 \\ \mathbf{0} & \boldsymbol{G} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{G} \\ \boldsymbol{G}^2 & 0 \end{bmatrix}$$

$$\boldsymbol{P}^4 = \boldsymbol{P}^3\,\boldsymbol{P} = \begin{bmatrix} 0 & \boldsymbol{G} \\ \boldsymbol{G}^2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} = \begin{bmatrix} \boldsymbol{G}^2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{G}^2 \end{bmatrix}$$

$$\boldsymbol{P}^5 = \boldsymbol{P}^4\,\boldsymbol{P} = \begin{bmatrix} \boldsymbol{G}^2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}^2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{G}^2 \\ \boldsymbol{G}^3 & 0 \end{bmatrix}$$

...... (5.13)

Substitute (5.13) into (5.12) to get:

$$e^{\boldsymbol{P}(t-t')} = \boldsymbol{P}^0 + \boldsymbol{P}(t-t') + \frac{\boldsymbol{P}^2}{2!}(t-t')^2 + \frac{\boldsymbol{P}^3}{3!}(t-t')^3 + \ldots\ldots$$

$$= \begin{bmatrix} 1 & 0 \\ \boldsymbol{0} & 1 \end{bmatrix} + (t-t')\begin{bmatrix} 0 & 1 \\ \boldsymbol{G} & 0 \end{bmatrix} + \frac{(t-t')^2}{2!}\begin{bmatrix} \boldsymbol{G} & 0 \\ \boldsymbol{0} & \boldsymbol{G} \end{bmatrix} + \frac{(t-t')^3}{3!}\begin{bmatrix} 0 & \boldsymbol{G} \\ \boldsymbol{G}^2 & 0 \end{bmatrix}$$

$$+ \frac{(t-t')^4}{4!}\begin{bmatrix} \boldsymbol{G}^2 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{G}^2 \end{bmatrix} + \frac{(t-t')^5}{5!}\begin{bmatrix} 0 & \boldsymbol{G}^2 \\ \boldsymbol{G}^3 & 0 \end{bmatrix} \ldots\ldots \quad (5.14)$$

Define

$$\boldsymbol{\omega}^2 \equiv -\boldsymbol{G} \quad (5.15)$$

and then Equation (5.14) becomes

$$e^{\boldsymbol{P}(t-t')} = \boldsymbol{P}^0 + \boldsymbol{P}(t-t') + \frac{\boldsymbol{P}^2}{2!}(t-t')^2 + \frac{\boldsymbol{P}^3}{3!}(t-t')^3 + \ldots\ldots$$

$$= \begin{bmatrix} 1 & 0 \\ \boldsymbol{0} & 1 \end{bmatrix} + (t-t')\begin{bmatrix} 0 & 1 \\ -\boldsymbol{\omega}^2 & 0 \end{bmatrix} + \frac{(t-t')^2}{2!}\begin{bmatrix} -\boldsymbol{\omega}^2 & 0 \\ \boldsymbol{0} & -\boldsymbol{\omega}^2 \end{bmatrix} + \frac{(t-t')^3}{3!}\begin{bmatrix} 0 & -\boldsymbol{\omega}^2 \\ \boldsymbol{\omega}^4 & 0 \end{bmatrix}$$

$$+ \frac{(t-t')^4}{4!}\begin{bmatrix} \boldsymbol{\omega}^4 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\omega}^4 \end{bmatrix} + \frac{(t-t')^5}{5!}\begin{bmatrix} 0 & \boldsymbol{\omega}^4 \\ -\boldsymbol{\omega}^6 & 0 \end{bmatrix} \ldots\ldots$$

$$= \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(t-t')^{2n}}{(2n)!}(-1)^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{2n}} & \boldsymbol{\omega}^{-\boldsymbol{1}} \sum_{n=0}^{\infty} \frac{(t-t')^{2n+1}}{(2n+1)!}(-1)^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{2n+1}} \\ -\boldsymbol{\omega} \sum_{n=0}^{\infty} \frac{(t-t')^{2n+1}}{(2n+1)!}(-1)^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{2n+1}} & \sum_{n=0}^{\infty} \frac{(t-t')^{2n}}{(2n)!}(-1)^{\boldsymbol{n}}\boldsymbol{\omega}^{\boldsymbol{2n}} \end{bmatrix}$$

$$= \begin{bmatrix} cos\,[(t-t')\boldsymbol{\omega}] & \boldsymbol{\omega}^{-\boldsymbol{1}} sin\,[(t-t')\boldsymbol{\omega}] \\ -\boldsymbol{\omega} sin\,[(t-t')\boldsymbol{\omega}] & cos\,[(t-t')\boldsymbol{\omega}] \end{bmatrix} \quad (5.16)$$

Substituting Equation (5.16) back to (5.11) with the definitions of Equation (5.4) and (5.7),

$$\boldsymbol{b} \equiv \begin{bmatrix} \boldsymbol{u} \\ \frac{\partial \boldsymbol{u}}{\partial t} \end{bmatrix} = \int_0^t \begin{bmatrix} cos\,[(t-t')\boldsymbol{\omega}] & \boldsymbol{\omega}^{-\boldsymbol{1}} sin\,[(t-t')\boldsymbol{\omega}] \\ -\boldsymbol{\omega} sin\,[(t-t')\boldsymbol{\omega}] & cos\,[(t-t')\boldsymbol{\omega}] \end{bmatrix} \begin{bmatrix} 0 \\ \boldsymbol{F} \end{bmatrix} dt'$$

we get:

$$\boldsymbol{u} = \int_0^t \boldsymbol{\omega}^{-\mathbf{1}} \sin[(t-t')\boldsymbol{\omega}]\, \boldsymbol{F}(t')dt' \tag{5.17}$$

With a long logistic of derivation based on mathematical physics on *Equation of Motion* in general curvilinear coordinates, we get a very simple close form solution of elastic vector wavefield as shown in Equation (5.17) solved via the method of propagator matrix. The elastic particle motion of the vector wave field $\boldsymbol{u}$ includes all the waveforms of elastic structure of the Earth and in our unconventional case, it describes all the elastic waveforms acquired from our 3C, 9C, VSP and micro seismic surveys.

As we can see from Equation (5.17), the convolution of the harmonic with density normalized source vector $\boldsymbol{F}$ is weighted by ***Kernel Matrix*** $\boldsymbol{\omega}$ which is in the dimension of frequency. $\boldsymbol{\omega}$ defines the spatial elastic properties and it represents the elastic eigen frequency modes in subsurface. The summation of the frequency eigen modes from the integration provides the various propagation interference pattern which forms various waveforms of spheroidal and toroidal.

Kernel Matrix $\boldsymbol{\omega}$ is the kernel for vector wavefield full waveform inversion. Perturbation on $\boldsymbol{\omega}$ sets up the generalized inversion of elastic wavefield. As we can see from its definition, it is highly none-linear. Therefore, any linear seismic inversion suffers a considerable assumptions which need to be investigated before applying.

In the next chapter, we will simplify the ***Kernel Matrix*** $\boldsymbol{\omega}$ in the unconventional deformation space characterized by *Jacobian* matrix from Equation (3.7)

$$\boldsymbol{J} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}$$

Chapter 6: Simplified Elastic Wave Kernel in Unconventional Deformation Space

In our unconventional land survey deformation space, Jacobian matrix is simplified as shown by Equation (3.7). For convenience, we rewrite it here:

$$\boldsymbol{J} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}$$

Also for convenience of simplify the spatial kernel operator, we put Equation (4.22) here:

$$\boldsymbol{\nabla} \bullet \boldsymbol{\tau_1} = [\tau_{11} \quad \tau_{12} \quad \tau_{13}] = \frac{1}{J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n}\Big[\frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{j2} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1 + \frac{\partial}{\partial \xi_2}\left(\tau_{12} J_{j1} J_{m3}\, d\xi_j d\xi_m\right) d\xi_2 + \frac{\partial}{\partial \xi_3}\left(\tau_{13} J_{j1} J_{m2}\, d\xi_j d\xi_m\right) d\xi_3\Big] = \frac{1}{J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n}\Big[\frac{\partial}{\partial \xi_1}\left(C_{11kl} u_{k,l} J_{j2} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1 + \frac{\partial}{\partial \xi_2}\left(C_{12kl} u_{k,l} J_{j1} J_{m3}\, d\xi_j d\xi_m\right) d\xi_2 + \frac{\partial}{\partial \xi_3}\left(C_{13kl} u_{k,l} J_{j1} J_{m2}\, d\xi_j d\xi_m\right) d\xi_3\Big]$$

Substituting the simplified *Jacobian* matrix elements into above equation, the first term becomes:

$$\frac{1}{J_{j1} J_{m2} J_{n3}\, d\xi_j d\xi_m d\xi_n} \frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{j2} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1 = \frac{1}{J_{11} J_{m2} J_{n3}\, d\xi_1 d\xi_m d\xi_n} \frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{12} J_{m3}\, d\xi_1 d\xi_m\right) d\xi_1 + \frac{1}{J_{21} J_{m2} J_{n3}\, d\xi_2 d\xi_m d\xi_n} \frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{22} J_{m3}\, d\xi_2 d\xi_m\right) d\xi_1 + \frac{1}{J_{31} J_{m2} J_{n3}\, d\xi_3 d\xi_m d\xi_n} \frac{\partial}{\partial \xi_1}\left(\tau_{11} J_{32} J_{m3}\, d\xi_j d\xi_m\right) d\xi_1$$

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